ARPA Network Information Center
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Packet Radio Temporary Note 6

Network Analysis Corp.
25 January 1973

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COMBINATORIAL ASPECTS OF MESSAGE FLOW IN PACKET RADIO NETS

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PART I

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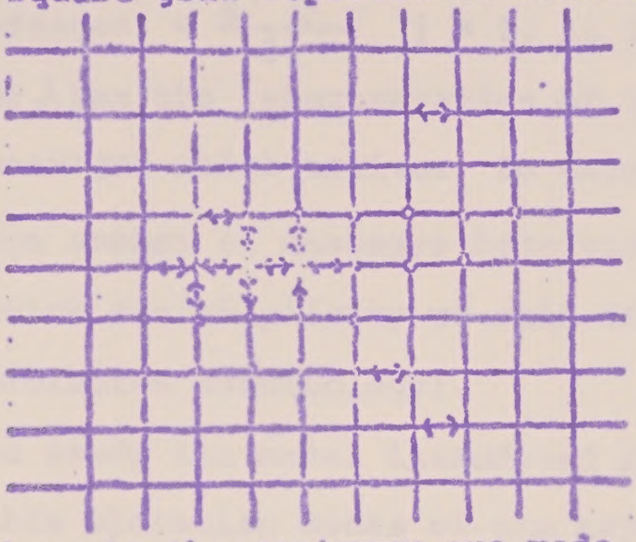
I. INTRODUCTION

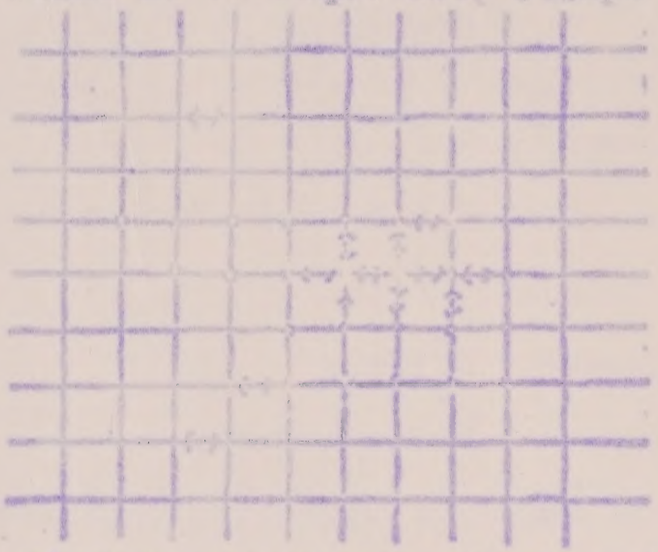
We introduce and analyze a model for the ARPA Packet Radio System. It is assumed that repeaters are located at the corner points of a square grid depicted as follows:



Unless assumptions to the contrary are made, a message received at any repeater is communicated (perfectly) to each of its four nearest neighbors, and received by those neighbors at the next time point.

Time is quantized in unit intervals (say one second). A packet that arrives within one interval is retransmitted within this interval. We specifically omit consideration of packet length, channel utilization, and other propagation properties and effects. Our goal is to study message flow assuming all electronics have been properly designed.

In section II we analyze the effects of a single message originating at any repeater at time t=0. No other messages are introduced. The explosive effects of the single message are studied.



In section III we assume that messages arrive independently at each node at each point in time accordingly to a Poisson distribution. That is, the probability that exactly j messages arrive at any node at each point in time is given by,

$$P \text{ (exactly } j \text{ messages)} = \frac{e^{-\lambda} \lambda^{j}}{j!} \qquad j = 0, 1, 2, \ldots,$$

the single parameter λ has the interpretation of the mean or average number of messages which arrive. In this case we determine the average number of messages received at a given point in the grid which for simplicity we call the origin. (from Cartesian coordinates terminology).

In section IV we study the model introduced in section III under various possible operating modes of the repeaters. A measure of efficiency of the modes is introduced and calculated for the four modes considered. Clearly each of the modes produce a reduction in redundant message flow. In section V we mix the operating modes and analyze the message flow.

The final section contains an analysis of the same model as considered in section I except that the grid is bounded and closed. The effect of a single message is determined by an algorithm for each point in time. Some conjectures on general solutions are offered.

II. PROPERTIES OF SINGLE MESSAGE PROPAGATION

We consider the infinite plane model and determine:

a) The number of repeaters which receive the message for the first time at time t: t = 0, 1, 2,....

b) The number of repeaters which have seen the single message by time t.

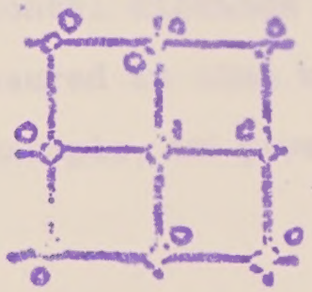
c) The number of copies of the single message received by any repeater at time t.

The assumptions are:

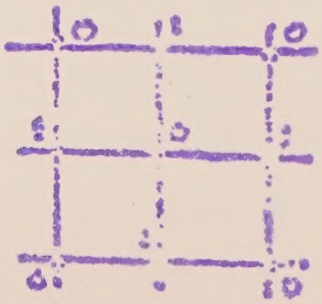
1) A single message arrives at a given node at time t = 0, no other messages are introduced into the network. We assume the message originated at the origin, (Cartesian coordinates (0,0)).

2) Message transmission is perfect, i.e., after one time unit each of the four neighbors to any repeater receive all messages transmitted by the repeater at the previous time point.

Some diagrams and numbers are helpful to fix ideas.



t = 0, (one message at origin, no messages elsewhere)



t = 1, (a single message at each of the four neighbors, no messages elsewhere)

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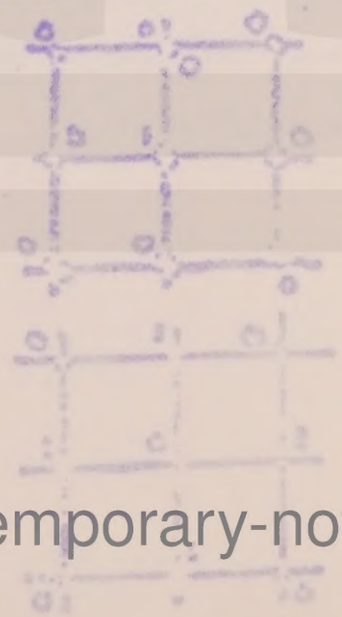

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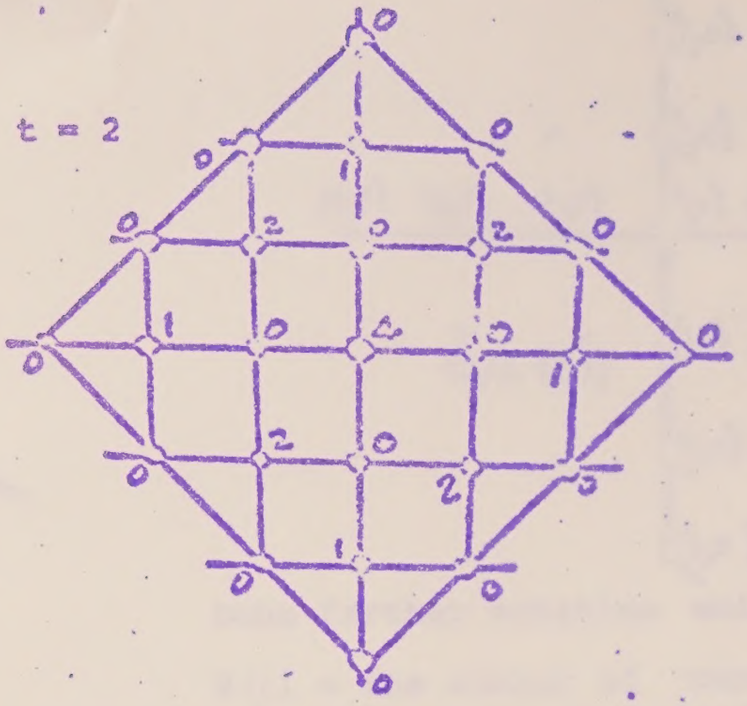

(4 messages at the origin,
0 Message at all repeaters
1 step from origin, 1 message
at each of four repeaters
2 steps in either the horizonatl
or vertical directions, 2 messages
at repeaters, 1 unit in horizontal
direction and 1 unit in vertical
direction.

By examining the diagrams we are led to introduce a coordinate system based on distance as measured in steps to reach a repeater and horizontal distance of the repeater from the origin. The quadrant symmetry of the model also indicates use of these coordinates.

The coordinates of a repeater are denoted by (d,j) where d is the distance of the repeater from the origin measured in minimum time units a message needs to arrive at the repeater from the origin; the second coordinate j is the horizontal distance of the repeater from the origin again measured in time units but only in the horizontal direction.

For example, we give some coordinates:

By examining the diagrams we are led to introduce a coordinate system based on distance as measured in steps to reach a repeater and horizontal distance of the repeater from the origin. The quadrant symmetry of the model also indicates use of these coordinates.

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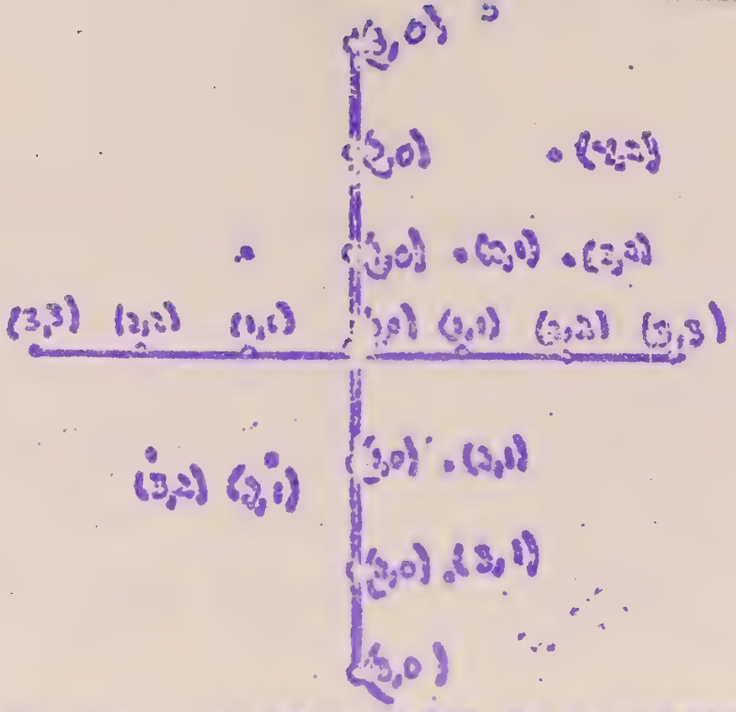


Some further notation which is necessary;

B(t) = the number of repeaters which receive the message for the first time at time t.

Clearly B (t) is the number of repeaters whose first coordinate is t. i.e. that are at distance t from the origin.

A(t) = the number of repeaters which have seen the message by time t. Clearly $A(t) = \sum_{j=0}^{t} B(j)$.

$N_j^d(t)$ = number of copies of the message received by a given repeater at coordinates (d,j) at time t. Clearly

a) $N_j^d(t) = 0$ for $d > t$

b) $N_j^d(d+2k+1) = 0$ for $k = 1, 2, \ldots$

Thus it is necessary to compute $N_j^d(d+2k)$ $k = 0, 1, 2, \ldots$

A) Calculation of B(t):

The quantity B(t), (the number of repeaters at distance t from the origin is the number of repeaters which receive the message for the first time t, is easy to compute. This quantity is given by the number of integer solutions to $|i| + |j| = t$. Since a repeater is at distance d from the origin if and only if its' Cartesian coordinates(i,j) satisfy

$$\sum_{j=0}^{t} B(j).$$

$$= 0 \text{ for } q > t$$

$$= 0 \text{ for } k = 1, 2, \ldots$$

$$N_j^q(q+2k) \quad k = 0, 1,$$

$B(t)$, (the number of repeaters

$|i| + |j| = t$, we can solve this equation and count solutions. Note that

$i=0$,	$j=t$	or $-t$	2 solutions
$i=1$,	$j=t-1$	or $-t+1$	2 solutions
$i=-1$,	$j=t-1$	or $-t+1$	2 solutions
$i=2$,	$j=t-2$	or $-t+2$	2 solutions
$i=-2$,	$j=t-2$	or $-t+2$	2 solutions
.	.	.	
.	.	.	
.	.	.	
$i=t-1$,	$j=1$	or $j=-1$	. . .
$i=-t+1$,	$j=1$	or $j=-1$	. . .
$i=t$,	$j=0$		
$i=-t$,	$j=0$		

The number of solutions is $B(0) = 1$

$$B(t) = 2 + 4(t-1) + 2 = 4t \text{ for } t \geq 1.$$

To compute $A(t)$, we sum $B(t)$ and obtain

$$A(t) = \sum_{j=0}^{t} B(j) = 1 + \sum_{j=1}^{t} 4(j) = 1 + 4\sum_{j=1}^{t} j = 1 + 4\,\frac{t(t+1)}{2}$$

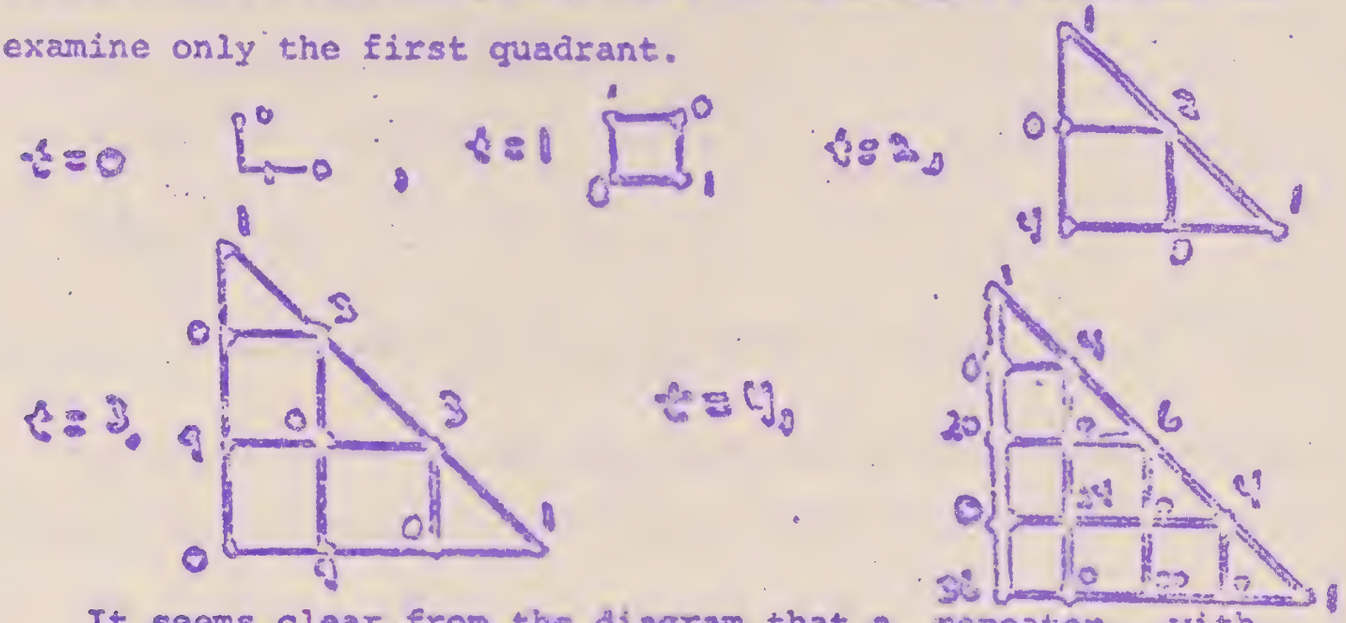
$$= 1 + 2t^2 + 2t = 2t^2 + 2t + 1$$

The rate at which $A(t)$ the number of repeaters which receive the message by time t grows as $A'(t) = 4t + 2$ which is linear in t.

B. The quantity $N_j^d(d + 2k)$, $k = 0, 1, 2, \ldots$

The quantity $N_q^j(q + 2k)$, $k = 0, 1, 2, \ldots$

To compute the number of copies of the message received at a repeater with coordinates (d,j) at time d + 2k, k = 0, 1, 2,... we first draw some diagrams. Due to symmetry it suffices to examine only the first quadrant.



It seems clear from the diagram that a repeater with coordinates (d,j) will receive at t=d the number of messages which is given by the binomial coefficient $\binom{d}{j}$. From the diagram we note the relationship of the outer edge to the d^{th} row of a Pascal triangle:

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

This result is also apparent from an argument based on the number of paths of a message from (0,0) to (d,j). The number of messages received at a repeater with coordinates (d,j) is given by the number of paths from (0,0) to (d,j) which is obviously $\binom{d}{j}$. To determine a general formula for $N_j^d(d + 2k)$ for $k > 1$, we can write and solve the appropriate difference equation.

$$N_j^d(d+2k) = N_{j-1}^{d-1}(d+2k-1) + N_{j+1}^{d+1}(d+2k-1)$$
$$+ N_j^{d+1}(d+2k-1) + N_j^{d-1}(d+2k-1)$$

for $k = 1, 2, \ldots; d = 0, 1, 2, \ldots; j = 0, 1, 2, \ldots, d$.

The initial conditions are

$N_j^d(t) = 0$ if $t < d$,

$N_j^d(d) = \binom{d}{j}$.

The solution to this equation is given by $N_j^d(d+2k) = \binom{d+2k}{k}\binom{d+2k}{k+j}$

To check its validity note that the initial conditions are satisfied, and apply the well known definition of binomial coefficients;

$$\binom{n}{j} = \binom{n-1}{j-1} + \binom{n-1}{j}, \qquad n \geq 1 \quad j \geq 1.$$

$$N_{j-1}^{d-1}(d-1+2k) + N_{j+1}^{d+1}(d+1+2(k-1)) + N_j^{d+1}(d+1+2(k-1)) + N_j^{d-1}(d-1+2k)$$

$$= \binom{d+2k-1}{k}\binom{d+2k-1}{k+j-1} + \binom{d+2k-1}{k-1}\binom{d+2k-1}{k+j} + \binom{d+2k-1}{k-1}\binom{d+2k-1}{k+j-1} + \binom{d+2k-1}{k}\binom{d+2k-1}{k+j}$$

$$= \binom{d+2k-1}{k}\left[\binom{d+2k-1}{k+j-1} + \binom{d+2k-1}{k+j}\right] + \binom{d+2k-1}{k-1}\left[\binom{d+2k-1}{k+j} + \binom{d+2k-1}{k+j-1}\right]$$

$$= \binom{d+2k-1}{k}\binom{d+2k}{k+j} + \binom{d+2k-1}{k-1}\binom{d+2k}{k+j}$$

$$= \binom{d+2k}{k+j}\left[\binom{d+2k-1}{k} + \binom{d+2k-1}{k-1}\right] = \binom{d+2k}{k+j}\binom{d+2k}{k}$$

For k very large with respect to d we can use a Stirling approximation to note the $N_j^d(d+2k) \sim 2^{4k}$ i.e. grows as 2^{4k}.

$$\binom{q+2k}{k}\binom{k+1}{k}+\binom{q+2k-1}{k-1}\binom{k+1}{k+1}$$

$$\binom{q+2k}{k}\binom{k+1}{k}+\binom{q+2k-1}{k-1}\binom{k+1}{k+1}$$

$$\binom{q+2k}{k}\binom{k+1}{k}+\binom{q+2k-1}{k-1}\binom{k+1}{k+1}$$

To summarize:

a) $B(t) = 4t$, $t \geq 1$ $B(0) = 1$,

b) $A(t) = 2t^2 + 2t+1$ $t \geq 0$,

c) $N_j^d(d+2k) = \binom{d+2k}{k+j}\binom{d+2k}{k} \sim 2^{4k}$ for large k .

III. POISSON INPUT AT EACH REPEATER

We analyze the same general model as in section II with new assumptions concerning origination of messages. As in section II, repeaters are located at corner points of the infinite square lattice and message transmission is perfect. Our new assumptions are:

1) At each point in time, starting at t=0, messages originate at each repeater according to a Poisson distribution with mean λ.

2) The arrivals (originations) at each repeater are independent over time and different repeaters.

The probability that exactly j new messages originate at any time point at any repeater is

$$\frac{e^{-\lambda}\lambda^{j}}{j!}, \quad j = 0, 1, 2, \ldots$$

We compute formulas for;

a) $N_o(t)$ = average number of messages which arrive at the origin at time t. Since all repeaters are statistically identical there is no loss in generality in studying message flow at the origin.

b) $N'_o(t)$ = Average number of distinct messages which arrive at the origin at time t for the first time.

c) Ieff(t)=Inefficiency of the network defined by;

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on the origin at time t for the first time.

d) $\bar{n}_j(t)$ = Average number of distinct messages which arrive at the origin at time t. Since all repeaters are statistically identical there is no loss in generality in assuming

that exactly 1 new message originates at any repeater in $(t-1, t)$.

Then $\bar{n}_j(t)$ satisfies the following recursive formulae for;

$\bar{n}_j(t)$ = average number of messages which arrive at the origin at time t for the first time.

$\bar{n}_j(t) = 0, \quad t < j$

$\bar{n}_j(t) = \sum_{i=1}^{t-j} \binom{t-j}{i} p^i (1-p)^{t-j-i} \left[\bar{n}_{j-1}(t-i) + 1 \right], \quad t \geq j$

$\bar{n}_0(t) = 0, \quad t \geq 0$

$\bar{n}_1(t) = 1 - (1-p)^t, \quad t \geq 1$

$\bar{n}_2(t) = 2 - 2(1-p)^t - t p (1-p)^{t-1}, \quad t \geq 2$

$\bar{n}_3(t) = 3 - 3(1-p)^t - 3tp(1-p)^{t-1} - \frac{t(t-1)}{2} p^2 (1-p)^{t-2}, \quad t \geq 3$

$\bar{n}_4(t) = 4 - 4(1-p)^t - 4tp(1-p)^{t-1} - 2t(t-1)p^2(1-p)^{t-2} - \frac{t(t-1)(t-2)}{6} p^3 (1-p)^{t-3}, \quad t \geq 4$

with mean λ.

(originating) each repeater

over time and different repeaters.

that exactly 1 new message originates

at any repeater in

$$Ieff(t) = \frac{No(t)}{N_o'(t)} = \frac{\text{average number of messages}}{\text{average number of new messages for the 1st time.}}$$

This is a measure of inefficiency since the larger Ieff, the more inefficient the system.

The actual number of messages which arrive at the origin at time t is a random variable. In fact it is a sum of a large number (when t is fairly large) of independent random variables. The summand random variables can loosely be described as the contribution to message flow at the origin arising out of some number of messages originating at each repeater at each point in time.

To compute $N_o(t)$ we can sum up all the contributions. This is interesting but tedious. A simpler method is to compute $X_o(t)$ which we define as the number of messages arriving at the origin at time t in a deterministic model obtained by assuming that at each point in time, at each repeater, exactly one new message originates. It will then follow that

$$N_o(t) = \lambda X_o(t).$$

Similarly, if we define $X_o'(t)$ to be the number of distinct messages that arrive at the origin at time t in the deterministic model it will follow that

$$N_o'(t) = \lambda X_o'(t).$$

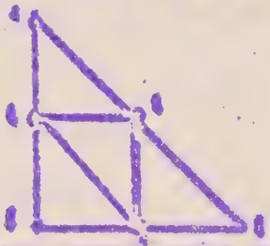
We indicate an armwaving proof of the first assertion. The quantity No(t) is a sum of average contribution to the flow at

the origin at time t, as a result of messages originating at repeaters less than t units in distance and times earlier than t. The average contribution from each repeater is a constant (not with time) at each fixed time point and repeater, multiplied by λ, the average generation rate. (the constant is given by the calculations in section II and depends on the coordinates (d,j) and time. Thus λ factors from the sum and $N_o(t)$ is λ multiplied by the total flow resulting from a single message originating at each repeater at each point in time.

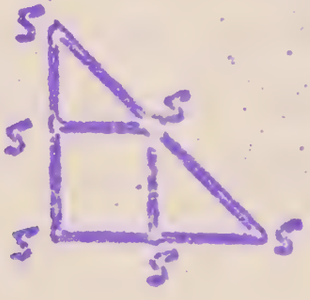
We now make the specific assumption.

At each point in time and at each repeater, a single new message is originated.

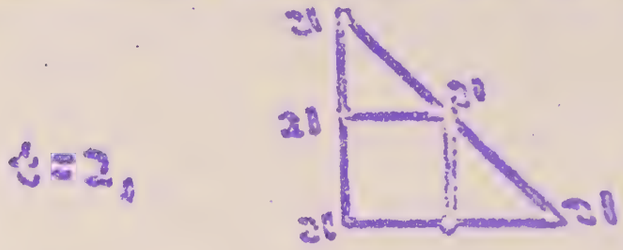
Under this model to compute $X_o(t)$ and hence $N_o(t)$ is trivial. To fix ideas we depict the situation at three time points



At time zero one message originates at each repeater, hence the message flow is $X_o(0)=1$.



At time 1 each repeater receives 5 messages, one from each of four nearest neighbors and one new message.



At time t = 2 each repeater receives 5 messages from each of its four nearest neighbors and one new message for a total of 21.

To compute $X_o(t)$ in general we note that each repeater is statistically identical in terms of message flow. Hence,

$$X_o(t) = 4X_o(t-1) + 1 \quad t \geq 1$$

$$X_o(0) = 1.$$

This difference equation is trivial to solve and hence,

$$X_o(t) = \frac{4^{t+1}-1}{3} \quad t \geq 0.$$

Thus

$$N_o(t) = \frac{\lambda}{3}(4^{t+1}-1)$$

To compute $N_o'(t)$ we consider the same deterministic model and compute $X_o'(t)$, the number of distinct messages which arrive at the origin for the first time at time t. It is easy to compute $X_o'(t)$ from the following table by summing contributions.

Time of Origination	Distance from Origin	Number of Messages
o	t	4t
1	t-1	4(t-1)
2	t-2	4(t-2)
t-1	1	4
t	0	1

The first column is the time the message first appeared in the system if it is received by the origin for the first time at time t. The second column indicates the distance from the origin that the message originated. The third column indicates the number of message originated at that time and distance which are received at the origin at time t. Thus,

$$X'_o(t) = 4t + 4(t-1) + \ldots + 4 + 1 = \sum_{j=1}^{t} 4j+1$$

$$= 2t^2 + 2t + 1, \quad t \geq 0$$

Thus,

$$N'_o(t) = \lambda(2t^2 + 2t + 1)$$

The inefficiency of this "undamped" network is

$$I_{eff}(t) = \frac{\frac{\lambda}{3}(4^{t+1}-1)}{\lambda(2t^2+2t+1)} = \frac{4^{t+1}-1}{3(2t^2+2t+1)} \sim \frac{4^{t+1}}{6t^2}$$

The inefficiency grows rapidly with time for this undamped system.

$$\frac{}{}$$

IV. ANALYSIS OF REPEATER OPERATING FEATURES

In this section we analyze the Poisson input model developed in the previous section. This analysis is focused on the message flow when new "hardware" or operating features of the repeaters inhibit message flow. We analyze the effects of each of four possible modes of operation.

A. No Message Can Be Transmitted More Than k Times

In this mode it is assumed that each message has a counting feature whereby each time the message is repeated the counter is updated by one unit. When the counter reaches the number k the message is no longer repeated and disappears from the system.

In this mode we compute;

a) $N_o(t)$ = average number of messages received by the origin at time t.

b) $N_o'(t)$ = average number of distinct messages received at the origin for the first time at time t.

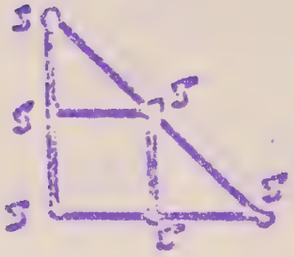
$$\text{Ieff}(t) = \frac{N_o(t)}{N_o'(t)}$$

Once again by the argument presented in the previous section it suffices to consider the deterministic model and to compute $X_o(t)$ and $X_o'(t)$. To fix ideas we diagram the first 6 time points, inherently assuming $k \geq 5$.

$t = 0$,

one message at each repeater,
all of age zero

$t = 5$,



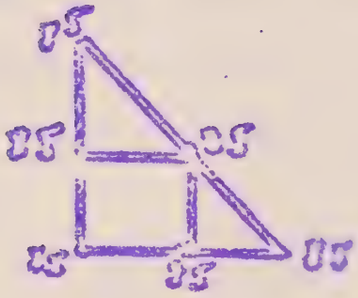
five messages each repeater:
one of age zero
four of age one

$t = 21$

21 messages at each repeater

1 of age zero
4 of age one
16 of age two

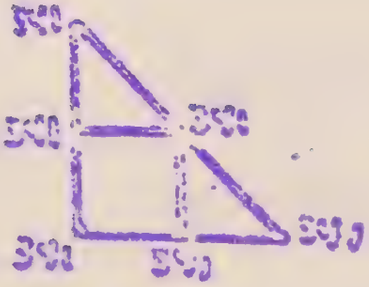
$t = 85$



85 messages at each repeater

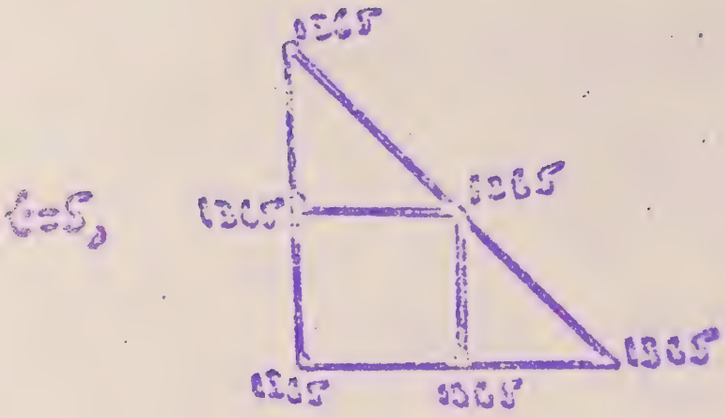
64 of age three
16 of age two
4 of age one
1 of age zero

$t = 341$



341 messages at each repeater

256 of age four
64 of age three
16 of age two
4 of age one
1 of age zero



1365 Messages	
1024	of age 5
256	of age 4
64	of age 3
16	of age 2
4	of age 1
1	of age 0
1365	TOTAL

Let $X_o^{>k}(t)$ be the number of messages arriving at the origin at time t whose age is less than k. That is, those that will be transmitted. Since the flow at all repeaters is statistically identical:

$$X_o(t) = 4X_o^{>k}(t-1) + 1 , \qquad t \geq 1 .$$

The number of messages received at the origin at time t is four times the number that will be transmitted by any of the four nearest neighbors plus one new one. Once again this is trivial difference equation to solve in k. We obtain as a solution:

$$X_o(t) = \frac{4^{k+1}-1}{3} \qquad t \geq k,$$

$$X_o(t) = \frac{4^{t+1}-1}{3} \qquad t \leq k .$$

Hence by the arguments in section III.

$$N_o(t) = \begin{cases} \lambda\left[\frac{4^{k+1}-1}{3}\right] & t \geq k \\ \lambda\left[\frac{4^{t+1}-1}{3}\right] & t < k, \end{cases}$$

A similar analysis shows that,

$$N_o'(t) = \begin{cases} 2k^2 + 2k + 1, & t \geq k, \\ 2t^2 + 2t + 1 & t \leq k. \end{cases}$$

Thus:

$$Ieff(t) = \frac{\frac{\lambda}{3}[4^{k+1}-1]}{\lambda(2k^2 + 2t + 1)} \qquad t \geq k.$$

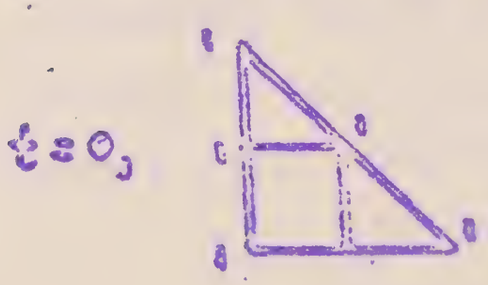
$$= \frac{4^{k+1}-1}{3(2k^2 + 2k + 1)} \sim \frac{4^{k+1}}{6k^2} = \frac{2 \cdot 4^k}{3k^2}.$$

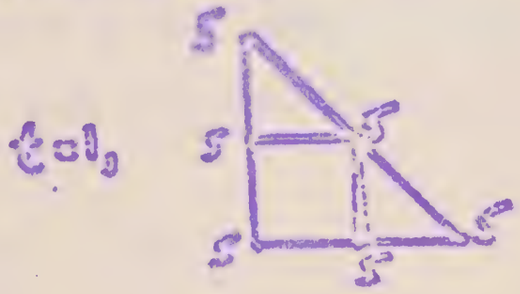
In tabular form for k=1, 2, 3, 4, 5 the inefficiencies are given

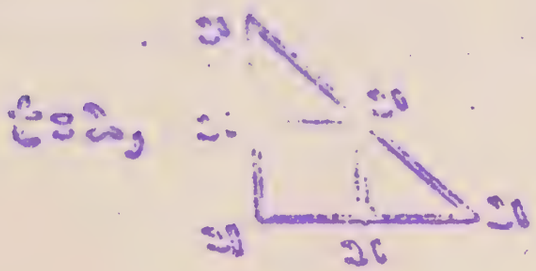
Ieff	1	1.62	3.40	8.3	22.4	....
k	1	2	3	4	5	....

B. If the Same Message Arrives From Different Sources Only one Transmission is Made.

In this mode of operation a repeater has the ability to compare messages which arrive at the same instant. We may consider this mode to be a "memory" type system of length of time "one unit" or instantaneous. We seek to compute $N_o(t)$ and $N_o'(t)$. Again we consider the deterministic model. To fix ideas let us examine some early time points and compute multiplicities. All repeaters in this case also have statistically identical flows.







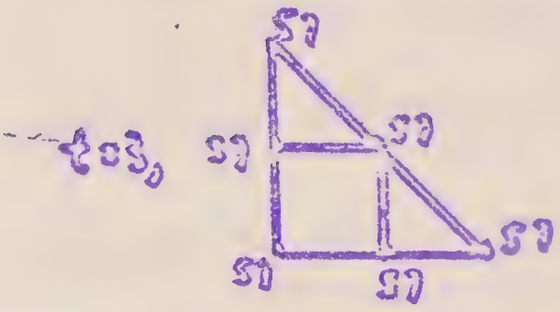
At time t = 2 all repeaters receive 21 messages, and they are statistically identical. Of the 21 total messages received at the origin only 14 are distinct.

We can partition this total using a table as follows:

Origin of Message	Time of Origin	Number Distinct
(0,0)	t=2	1
(1,1), (1,0)	t=1	4
(2,0), (2,1), 2,2)	t=0	8
(0,0)	t=0	1

The total distinct is 14,

Hence the diagram for t=3 is,



of the 57 = 4·14 +1, one is new and 14 came from each of four nearest neighbors.

The same method can be used to compute $X_o(2t)$ and $X_o(2t-1)$ in general. Let $X_o^d(2t)$ and $X_o^d(2t-1)$ be the number of distinct messages received at the origin at time (2t) and (2t-1) respectively. Clearly, these quantities satisfy:

$$X_o(2t) = 4X_o^d(2t-1) + 1 \quad \text{and}$$

$$X_o(2t+1) = 4X_o^d(2t) + 1.$$

Thus it suffices to compute $X_o^d(2t)$ and $X_o^d(2t-1)$.

Time of Error	
t=2	
t=1	
t=0	
t=0	

for t=3 is

of the 27 = 18, +1, one

and 14 same from each

used to compute X_d^o (2t) and

(2t) and X_d^o (2t-1) be the number

at the origin at time (2t) and (2t-1)

$$\text{and} \quad (2t-1) \; +1$$

$$(2t-1) \; +1.$$

Thus it remains to write X_j^i

To compute $x_o^d(2t)$ and $x_o^d(2t-1)$ we decompose each as follows:

$$x_o^d(2t) = x_o^{d,1}(2t) + x_o^{d,2}(2t) + x_o^{d,3}(2t) + \ldots + x_o^{d,t+1}(2t),$$

$$x_o^d(2t-1) = x_o^{d,1}(2t-1) + x_o^{d,2}(2t-1) + \ldots + x_o^{d,t}(2t-1).$$

Where $x_o^{d,v}(t)$ = number of distinct messages received at the origin at time t for the v^{th} time. The quantities $x_o^{d,v}(t)$ are computed by the following table, which essentially decomposes $x_o^{d,v}(t)$ by distance and time of origination of each message contributing to $x_o^{d,v}(t)$. For $x_o^d(2t-1)$:

1st time -	time of origin.	0	1	...	2t-2	2t-1
	dist. at origin.	2t-1	2t-2	...	1	0
2nd time -	time of origin.	0	1	...	2t-4	2t-3
	dist. of origin.	2t-3	2t-4	...	1	0
...						
(t-1)th time -	Time of origin.	0	1	2	3	
	dist. at origin.	3	2	1	0	
tth time -	time of origin.	0	1			
	dist at origin.	1	0			

The grand sum is;

$$x_o^{d,1} = 1 + 4\sum_{j=1}^{2t-1}(2t-j), \quad x_o^{d,2} = 1 + 4\sum_{j=3}^{2t-1}(2t-j), \ldots, x_o^{d,t} = 4\sum_{j=2t-1}^{2t-1}(2t-j)$$

since the number of messages originating at distance d is 4d as we have seen earlier. Therefore,

$$x_o^d(2t-1) = t + 4\sum_{k=1}^{t}\sum_{j=2k-1}^{2t-1}(2t-j)$$

$$X_q^o(2t-1) = t+4 \sum_{k=2}^{t} \sum_{j=2k-1}^{2t-1} (2t-j)$$

After summing we obtain:

$$X_o^d(2t-1) = \frac{t(4t+1)(2t+1)}{3},$$

By a similar analysis we can show that

$$X_o^d(2t) = \frac{(4t+3)(2t+1)(t+1)}{3},$$

Thus,

$$X_o(2t) = 4 \cdot X_o^d(2t-1) + 1 = \frac{4t(4t+1)(2t+1)}{3} + 1, \text{ or}$$

$$X_o(2t) = \frac{(4t+3)(8t^2+1)}{3}.$$

Similarly,

$$X_o(2t+1) = 4 \cdot X_o^d(2t) + 1$$

$$= \frac{4(4t+3)(2t+1)(t+1)}{3} + 1$$

$$= \frac{32t^3+72t^2+52t+15}{3}$$

Hence: $N_o(2t) = \frac{\lambda(4t+3)(8t^2+1)}{3}$,

$$N_o(2t+1) = \frac{\lambda(32t^3+72t^2+52t+15)}{3},$$

$$N_o^1(2t) = \lambda(2t^2+2t+1) = \lambda(8t^2+4t+1).$$

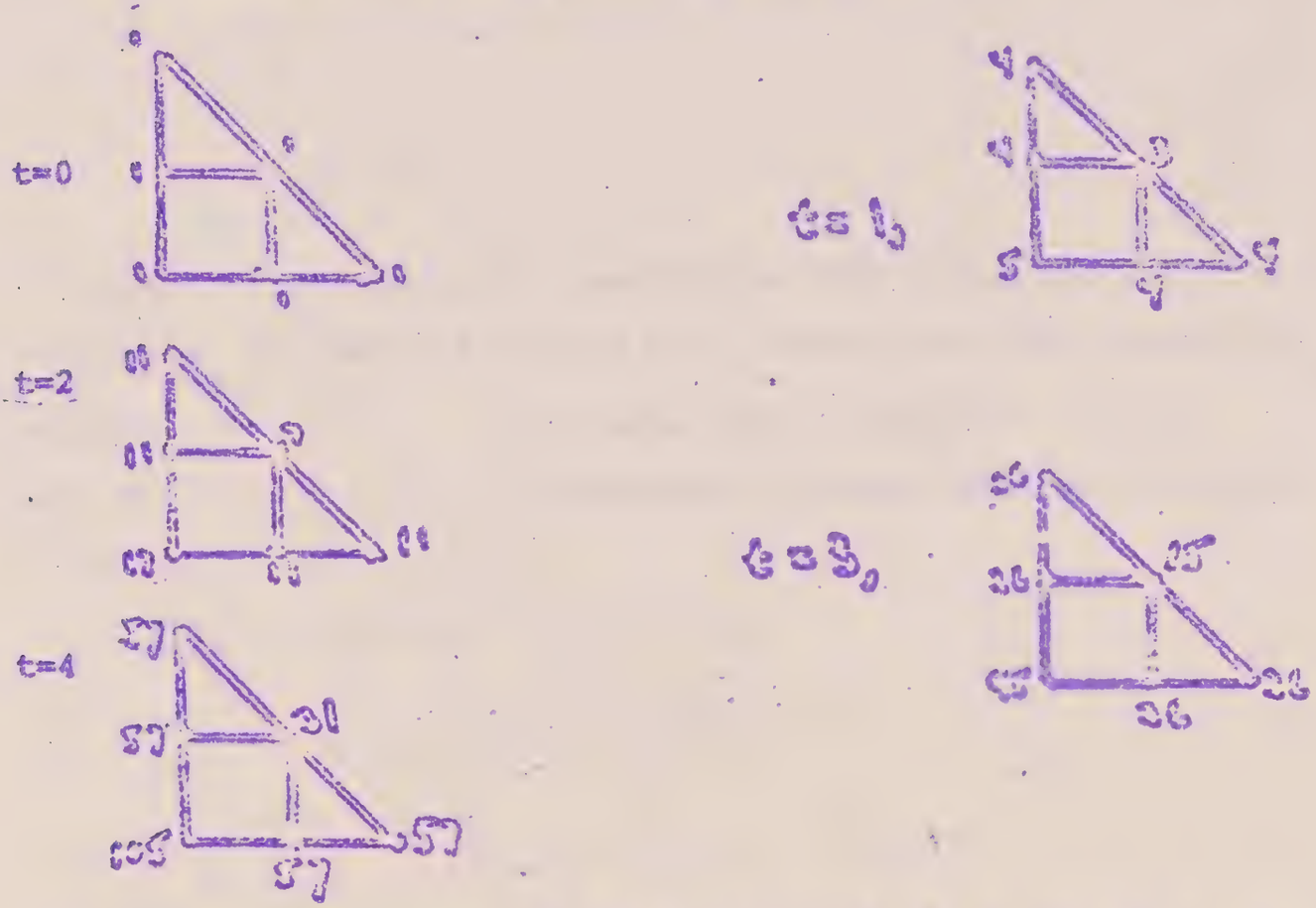
The efficiency of this mode is:

$$\text{Eff}(2t) = \frac{(4t+3)(8t^2+1)}{3(8t^2+4t+1)} \sim \frac{4t}{3}.$$

C. All Messages Are Transmitted In The Direction of a Fixed Station

In this section we study the effects of a directional system of operation. Once again, we assume independent Poisson input but study the deterministic case with no loss of generality. The assumption concerning direction is that when a message is transmitted at a repeater at distance d, the transmission is to all neighboring repeaters at distance d-1 from the origin.

Pictorially, the propagation is as follows:



Let $X_o(t)$ be the number of messages received at time t in the deterministic case.

Let $X_j^d(t)$ be the number of messages received at a repeater at distance d, horizontal distance j (i.e. with coordinates (d,j)) at time t.

We have immediately

$$X_o(t) = 4\ X_o^1(t-1) + 1, \qquad t \geq 1, \quad \text{and}$$

$$X_o^1(t) = 2\ X_1^2(t-1) + X_o^2(t-1) + 1, \quad \text{where}$$

$$X_1^2(t) = 2\ X_1^2(t-1) + 1$$

$$= 2\ X_1^2(t-2) + 2 + 1$$

$$=$$

$$= 2^{t+1} - 1 .$$

This follows from the observation that repeaters at horizontal distance zero or d will receive the same number of messages for $d > 0$. Repeaters with coordinates (d,j), $j = 1, 2, \ldots, d-1$ will also receive the same number of messages.

Thus;

$$X_o^1(t) = 2(2^t - 1) + X_o^1(t-1) + 1$$

$$= X_o^1(t-2) + 2^t + 2^{t+1} - 2$$

$$=$$

$$= 1 + 2^2 + 2^3 + \ldots + 2^{t+1} - t.$$

We conclude that,

$$Xo(t) = 2^{t+3} - 4t - 7 \qquad t = 0, 1, 2, \ldots.$$

Thus Xo(t) the expected number of messages received is $\frac{1}{4}(2^{t+3} - 4t - 7)$.

$$X_j^{\circ}(t-T) + T$$

$$z_t + z_{t+1}^{-z}$$

$$+z_{t+T}^{-t},$$

$$t=0,\ 1,\ 2,\ldots$$

$$-t-z$$

$$-t-T).$$

The expected number of distinct messages received is $\lambda(2t^2+2t+1)$.

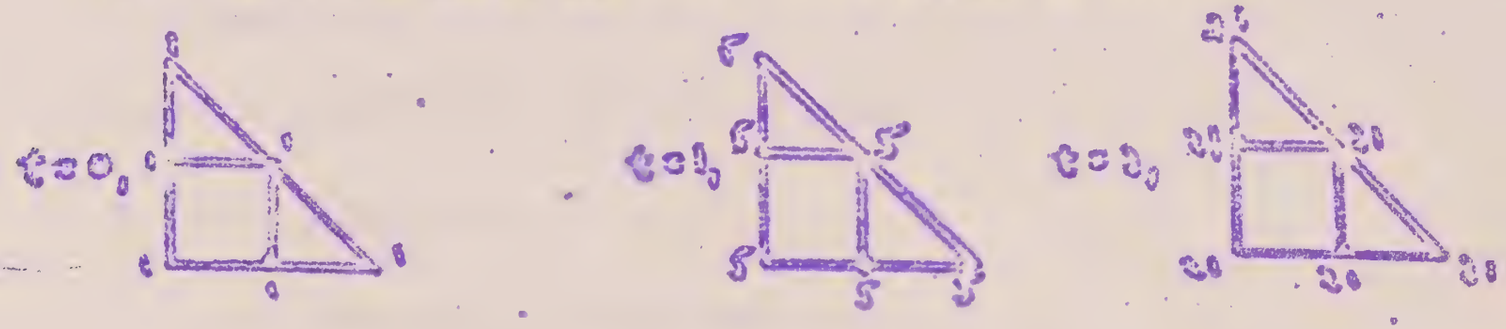
Therefore

$$Ieff(t) = \frac{No(t)}{N_o^1(t)} = \frac{2^{t+3}-4t-7}{2t^2+2t+1} \sim \frac{2^{t+3}}{2t^2} = \frac{2^{t+2}}{t^2}$$

D. A Repeater Never Transmits The Same Message More Than Once Except Upon Initial Reception

This mode of operation implies infinite but not instantaneous memory in the repeater. We analyze the deterministic case with no loss of generality.

Pictorially the first few time points appear as follows;



Of the 21 messages received at t=2 there are 17 which are received for the first time. These can be enumerated by point and time of origin: 1 new one at(t=2,d=0),4 at(t=1, d=1),12 at (t=0, d=2). Thus there will be 69 messages received at each repeater at t=3.

In general let:

$X_o(t)$ be the number of messages received at the origin at time t, and $X_o^1(t)$ be the number of messages received at the origin for the first time at time t. Then,

$$X_o(t) = 4\, X_o^1(t-1) + 1 .$$

If a message was received for the first time at any repeater at time t-1 it originated at distance t-1 at time 0, or at d=t-2 at time 1 or, ..., at distance 0 at time t-1. Summing these by their appropriate multiplicities we obtain,

$$X_o^1(t-1) = 4 \sum_{k=2}^{t} \sum_{j=0}^{t-k} \binom{t-k+1}{0} \binom{t-k+1}{j} + 1 ,$$

or,

$$X_o^1(t-1) = 2^{t+2} - 4t - 3 \qquad \text{and,}$$

$$Xo(t) = 2^{t+4} - 16t - 11 .$$

Similarly;

$$No(t) = \lambda (2^{t+4} - 16t - 11) \quad \text{and as before}$$

$$N_o^1(t) = \lambda (2t^2 + 2t + 1)$$

The efficiency of this mode is:

$$Eff = \frac{No(t)}{No^1(t)} = \frac{2^{t+4} - 16t - 11}{2t^2 + 2t + 1} \sim \frac{2^{t+3}}{t^2} .$$

V. Mixtures of Operating Features

In this section we analyze the effect of mixing the four operating features of repeaters on the efficiency of network operation.

A. Mixture of Modes B and D,

In this mixed mode a repeater has infinite memory for comparison of messages and instant memory. The result (operational) of this mixed mode is that when multiple reception of the same message is received only one transmission is made. Furthermore if a message is received for the second time it is not transmitted at all. The result of this combined mode is to reduce the transmission in the instantaneous mode to only messages received for the first time.

Therefore:

$$X_o(t) = 4X_o^{',d}(t-1) + 1$$

where

$X_o^{',d}(t)$ is the number of distinct messages received at the origin for the first time at time t. This quantity has been previously computed to be,

$$X_o^{',d}(t-1) = 2(t-1)^2 + 2(t-1) + 1 = 2t^2 - 2t - 1.$$

It follows that,

$$X_o(t) = 8t^2-8t-3, \text{ and}$$

$$N_o(t) = \lambda(8t^2-8t-3).$$

The efficiency of this mixed mode is seen to be

$$Eff(t) = \frac{\lambda(8t^2-8t-3)}{\lambda(2t^2+2t+1)} \quad \frac{8t^2-8t-3}{2t^2+\ldots} \sim 4$$

B. Mixture of Modes A and B

In this mixture of modes no message can be transmitted more than k-times and each repeater has instantaneous memory but not finite memory. This mode also provides an upper bound to the case where each repeater has zero capture and messages are dropped after k transmissions.

A little analysis will show that for $t \leq k$ there is no change in the message flow from the instantaneous memory case. For $t \geq k$ the exact same analysis as in the instantaneous memory only case shows that the formulas are exactly those except that t should be replaced by k.

Therefore in this mixed mode, for memory of k.

$$X_o(2t) = \frac{(2k+3)(8k^2+1)}{3}, \quad t \geq \left[\frac{k}{2}\right]$$

$$= \frac{(2t+3)(8t^2+1)}{3}, \quad t \leq \left[\frac{k}{2}\right]$$

$$X_o(2t+1) = \frac{32k^3+72k^2+52k+15}{3} \qquad t \geq k$$

$$= \frac{32t^3+72t^2+52t+15}{3} \qquad t \leq k.$$

Furthermore,

$$N_o(2t) = X_o(2t),$$

$$N_o(2t+1) = X_o(2t+1).$$

We arrive at the same result with t replaced by k.

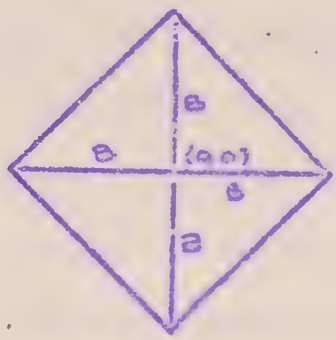
$$\text{Eff}(2t) = \frac{(4k+3)(8k^2+1)}{3(8k^2+4k+1)} \sim \frac{4}{3}k.$$

In tabular form;

Eff	1.33	2.07	3.0	3.7	3.87	3.9

VI. A CLOSED BOUNDARY MODEL

In this part of the model it is assumed that each repeater is less than B units from the origin. As in earlier cases, the units are measured in steps. The region is now closed and appears as follows:



Purthermore, in the initial stages of this model, it is assumed that a single message originates at the origin at time t=0. All message flow is generated as a result of this single message. Unfortunately, this model causes a loss of symmetries which facilitated the ease of obtaining closed form solutions for message flow in the previous models. However, some closed form analysis can be obtained. In particular it is not difficult to obtain an algorithm which will supply a complete analysis of message flow at any point in time on any repeater or station.

Let $N^{d}_{B,j}(2t+d)$ be the number of messages received at a station or repeater which is d-units from the origin and j-units from the y-axis at time (2t+d) where d=0, 1, 2,...,B-1, j=0, 1, 2,...B.

The equations for $N^{d}_{B,j}(2t+d)$ are identical to the equations for the open boundary case when $0 \leq t \leq B-d$ since the closing of the boundary does not affect message flow at a repeater at distance B-d until t=B+d. Therefore;

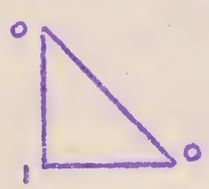
$$N^{d}_{B,j}(2t+d) = \binom{2t+d}{t}\binom{2t+d}{t+j}; \quad t \leq \frac{B}{2}.$$

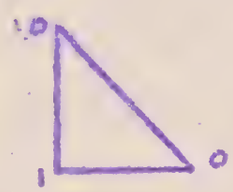
We can use this equation to compute message flow at all time points up to and including t=B. To compute message flow after time B, we can work backwards from the boundary and successively compute message flow at each station or repeater for any time point using the equation we will develop. First however, it is helpful to examine a particular case with a diagram and compare the closed boundary with the open boundary. We let B=4.

Closed Boundary at B=4 Open Boundary

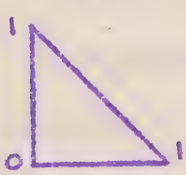
The closing of the boundary will not affect any repeaters or stations until t=6.

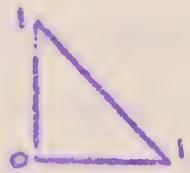
t=0



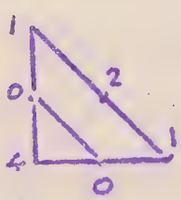


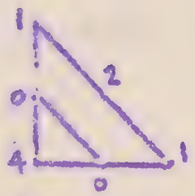
t=1



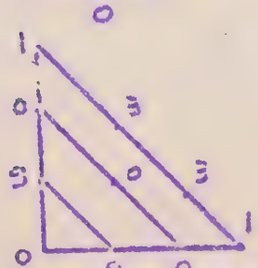


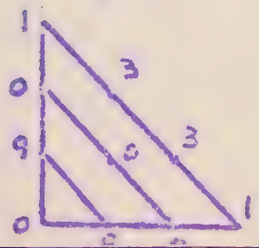
t=2

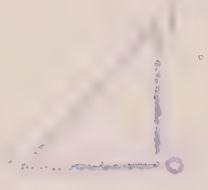


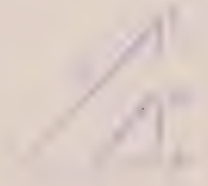


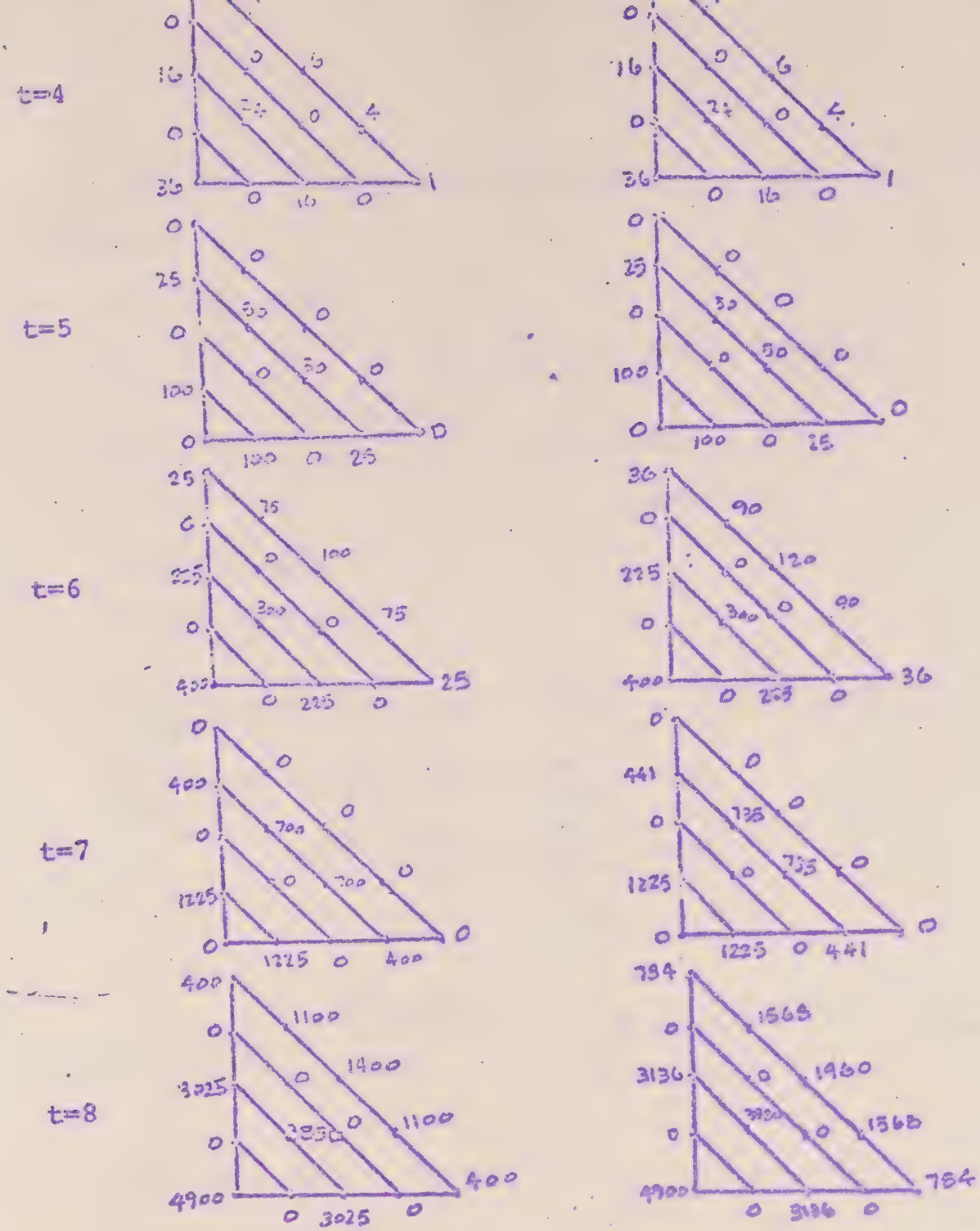
t=3











It is obvious how to continue for as long as one wishes to obtain the message flow. Since all message flow can be computed by the formulas of the preceeding section for $t \leq B$, the message flow with closed boundary can be computed by backward iteration for all times larger than B. The initial conditions at time t=B are given by the following relations,

At time t=B

$$N^{d}_{k,j}(B) = \begin{cases} \binom{B}{k}\binom{B}{k+j} & \text{if } B=2k+d \\ 0 & \text{otherwise.} \end{cases}$$

since $N^{d}_{B,j}(B)$ is zero unless B and d are both odd or even numbers.

At time t=B+1

$N^{B}_{B,j}(B+1)=0$, i.e. no messages at the boundary,

$$N^{d}_{B,j}(B+1)=\begin{cases}\binom{B+1}{k}\binom{B+1}{k+j} & \text{if } B=2k+d-1,\\ 0 & \text{otherwise.}\end{cases}$$

At time t=B+2

On the boundary:

$$N^{B}_{B,0}(B+2) = N^{B-1}_{B,0}(B+1) = \binom{B+1}{1}^2 = \binom{B+1}{B}^2 = N^{B}_{B,B}(B+2).$$

$$N^{B}_{B,j}(B+2) = N^{B-1}_{B,j}(B+1) + N^{B-1}_{B,j-1}(B+1)$$

$$=\binom{B+1}{B}\binom{B+2}{j+1} = N^{B}_{B,B-j}(B+2)$$

for $j=1, 2,\ldots,B-1$.

for $d<B$

$$N^{d}_{B,j}(B+2) =\begin{cases}\binom{B+2}{k}\binom{B+2}{k+j} & \text{if } B=2k+d-2,\\ 0 & \text{otherwise}\end{cases} \qquad j=0, 1, 2,\ldots,d.$$

At time t=B+3

On the boundary

$N^{B}_{B,j}(B+3) = 0$: $j=0, 1, 2,\ldots,B$.

At distance d=B-1

$$N^{B-1}_{B,0}(B+3) = N^{B}_{B,0}(B+2) + N^{B-2}_{B,0}(B+2) + 2N^{B}_{B,1}(B+2)$$

$$=\left[\binom{B+1}{B}+\binom{B+2}{B}\right]^2 = N^{B-1}_{B,B-1}(B+3).$$

$$N^{B-1}_{B,j}(B+3) = N^{B-2}_{B,j-1}(B+2) + N^{B}_{B,j+1}(B+2) + N^{B-2}_{B,j}(B+2)$$

$$+ N^{B}_{B,j}(B+2)$$

$$= \binom{B+3}{j+2}\left[\binom{B+1}{B} + \binom{B+2}{B}\right] = N^{B-1}_{B,B-1-j}(B+3).$$

$$j=1, 2, \ldots, B-2.$$

<u>At distance $d < B-1$</u>

$$N^{d}_{B,j}(B+3) = \begin{cases} \binom{B+3}{k}\binom{B+3}{k+j} & B=2k+d-3 \\ 0 & \text{otherwise} \end{cases} \qquad j=0, 1, \ldots, d.$$

<u>At time $t=B+4$</u>

On the boundary

$$N^{B}_{B,0}(B+4) = N^{B-1}_{B,0}(B+3) = \left[\binom{B+1}{B}+\binom{B+2}{B}\right]^2 = N^{B}_{B,B}(B+4).$$

$$N^{B}_{B,1}(B+4) = N^{B-1}_{B,0}(B+3) + N^{B-1}_{B,1}(B+3)$$

(Note that $N^{B}_{B,1}(B+4)$ must be computed separately since $N^{B-1}_{B,0}(B+3)$ and $N^{B-1}_{B,j}(B+3)$ for $j > 1$ do <u>not</u> share a common formula, i.e. are not special cases of a general formula.)

$$N^{B}_{B,1}(B+4) = N^{B}_{B,B-1}(B+4) = \left[\binom{B+1}{B}+\binom{B+2}{B}\right]\left[\binom{B+1}{B}+\binom{B+2}{B}+\binom{B+3}{B}\right]$$

$$N^{B}_{B,j}(B+4) = N^{B-1}_{B,j-1}(B+3) + N^{B-1}_{B,j}(B+3)$$

(Note the formula is the same for $j=1$ as above.)

$$= \left[\binom{B+1}{B}\binom{B+2}{B}\right]\left[\binom{B+4}{j+2}\right] = N^{B}_{B,B-j}(B+4)$$

$$j=2, \ldots, B-2.$$

<u>At distance $d=B-2$</u>

$$N^{B-2}_{B,0}(B+4) = N^{B-3}_{B,0}(B+3) + N^{B-1}_{B,0}(B+3) + 2\,N^{B-1}_{B,1}(B+3)$$

$$= \left[\binom{B+1}{B} + \binom{B+2}{B} + \binom{B+3}{B}\right]^2 = N^{B-2}_{B,B-2}(B+4)$$

$$N^{B-2}_{B,j}(B+4) = N^{B-3}_{B,j-1}(B+3) + N^{B-1}_{B,j+1}(B+3) + N^{B-1}_{B,j}(B+3) + N^{B-3}_{B,j}(B+3).$$

$$= \binom{B+4}{j+3}\left[\binom{B+1}{B} + \binom{B+2}{B} + \binom{B+3}{B}\right]$$

$$= N^{B-2}_{B,B-j-2}(B+4).$$

for $d < B-2$

$$N^{d}_{B,j}(B+4) = \begin{cases} \binom{B+4}{k}\binom{B+4}{k+j} & \text{if } B=2k+d-4 \\ 0 & \text{otherwise.} \end{cases}$$

At time $t=B+5$

at distance $d=B-1$

$$N^{B-1}_{B,0}(B+5) = N^{B-2}_{B,0}(B+4) + N^{B}_{B,0}(B+4) + 2\,N^{B}_{B,1}(B+4)$$

$$= \left[2\binom{B+1}{B} + 2\binom{B+2}{B} + \binom{B+3}{B}\right]^2 = N^{B-1}_{B,B-1}(B+5)$$

$$N^{B-1}_{B,1}(B+5) = N^{B-2}_{B,0}(B+4) + N^{B}_{B,2}(B+4) + N^{B-2}_{B;1}(B+4) + N^{B}_{B,1}(B+4)$$

$$= \left[2\binom{B+1}{B} + 2\binom{B+2}{B} + \binom{B+3}{B}\right]$$

$$\cdot \left[\binom{B+1}{B} + \binom{B+2}{B} + \binom{B+3}{B} + \binom{B+4}{B}\right] = N^{B-1}_{B,B-1}(B+5).$$

$$N^{B-1}_{B,j}(B+5) = N^{B-2}_{B,j-1}(B+4) + N^{B}_{B,j+1}(B+4) + N^{B-2}_{B,j}(B+4) + N^{B}_{B,j}(B+4)$$

$$= \binom{B+5}{j+3}\left[2\binom{B+1}{B} + 2\binom{B+2}{B} + \binom{B+3}{B}\right]$$

$$= N^{B-1}_{B,B-j-1}(B+5).$$

$j=2,\ldots,B-3.$

At distance B-3

$$N^{B-3}_{B,0}(B+5) = N^{B-4}_{B,0}(B+4) + N^{B-2}_{B,0}(B+4) + 2N^{B-2}_{B,1}(B+4)$$

$$= [\binom{B+1}{B} + \binom{B+2}{B} + \binom{B+3}{B} + \binom{B+4}{B}]^2 = N^{B-3}_{B,B-3}(B+5).$$

$$N^{B-3}_{B,j}(B+5) = N^{B-4}_{B,j-1}(B+4) + N^{B-2}_{B,j+1}(B+4)$$

$$+N^{B-4}_{B,j}(B+4) + N^{B-2}_{B,j}(B+4)$$

$$= \binom{B+5}{j+4}[\binom{B+1}{B} + \binom{B+2}{B} + \binom{B+3}{B} + \binom{B+4}{B}]$$

$$= N^{B-3}_{B,B-j-3}(B+5) \qquad j=1, 2, \ldots, B-4.$$

for $d < B-3$

$$N^{d}_{B,j}(B+5) = \begin{cases} \binom{B+5}{k}\binom{B+5}{k+j} & \text{if } B = 2k+d-5 \\ 0 & \text{otherwise.} \end{cases}$$

$j=0, 1, 2, \ldots, d.$

We will continue for three more time points to get some idea as to how the solution behaveswith time. We will omit the general equations and give only the results since it is clear that the expressions become too cumbersome for easy comprehension.

For time t=B+6

at the boundary

$$N^{B}_{B,0}(B+6) = [2\binom{B+1}{B} + 2\binom{B+2}{B} + \binom{B+3}{B}]^2 = N^{B}_{B,B}(B+6)$$

$$N^{B}_{B,1}(B+6) = [2\binom{B+1}{B} + 2\binom{B+2}{B} + \binom{B+3}{B}][3\binom{B+1}{B} + 3\binom{B+2}{B} + 2\binom{B+3}{B} + \binom{B+4}{B}$$

$$N^{B}_{B,2}(B+6) = [2\binom{B+1}{B} + 2\binom{B+2}{B} + \binom{B+3}{B}][\binom{B+1}{B} + \binom{B+2}{B} + \cdots + \binom{B+5}{B}]$$

$$N^{B}_{B,j}(B+6) = [2\binom{B+1}{B} + 2\binom{B+2}{B} + \binom{B+3}{B}][\binom{B+6}{J+3}]$$

$$j=3, 4, \ldots, B-3.$$

At distance $d=B-2$

$$N^{B-2}_{B,0}(B+6) = [3\binom{B+1}{B} + 3\binom{B+2}{B} + 2\binom{B+3}{B} + \binom{B+4}{B}]^2 = N^{B-2}_{B,B-2}(B+6)$$

$$N^{B-2}_{B,1}(B+6) = [3\binom{B+1}{B} + 3\binom{B+2}{B} + 2\binom{B+3}{B} + \binom{B+4}{B}]$$

$$\cdot [\binom{B+1}{B} + \binom{B+2}{B} + \cdots + \binom{B+5}{B}] = N^{B-2}_{B,B-3}(B+6)$$

$$N^{B-2}_{B,j}(B+6) = \binom{B+6}{j+4}[3\binom{B+1}{B} + 3\binom{B+2}{B} + 2\binom{B+3}{B} + \binom{B+4}{B}] = N^{B-2}_{B,B-2-j}(B+6)$$

$$j=2, \ldots, B-4.$$

At distance $d=B-4$

$$N^{B-4}_{B,0}(B+6) = [\binom{B+1}{B} + \binom{B+2}{B} + \cdots + \binom{B+5}{B}]^2 = N^{B-4}_{B,B-4}(B+6)$$

$$N^{B-4}_{B,j}(B+6) = \binom{B+6}{j+5}[\binom{B+1}{B} + \binom{B+2}{B} + \cdots + \binom{B+5}{B}]^2 = N^{B-4}_{B,B-4-j}(B+6)$$

$$j=1, 2, \ldots, B-5.$$

For $d < B-4$

$$N^{d}_{B,j}(B+6) = \begin{cases} \binom{B+6}{k}\binom{B+6}{k+j} & \text{if } B=2k+d-6 \\ 0 & \text{otherwise.} \end{cases}$$

At time $t=3+7$

At distance $d=B-1$

$$N^{B-1}_{B,0}(3+7) = [5\binom{B+1}{B} + 5\binom{B+2}{3} + 3\binom{B+3}{B} + \binom{B+4}{B}]^2 = N^{B-1}_{3,B-1}(3+7).$$

$$N_{B,1}^{B-1}(B+7) = [5\binom{B+1}{B} + 5\binom{B+2}{B} + 3\binom{B+3}{B} + \binom{B+4}{B}]$$
$$\cdot [4\binom{B+1}{B} + 4\binom{B+2}{B} + 3\binom{B+3}{B} + 2\binom{B+4}{B} + \binom{B+5}{B}]$$
$$= N_{B,B-2}^{B-1}(B+7).$$

$$N_{B,2}^{B-1}(B+7) = (\sum_{j=1}^{6}\binom{B+j}{B})\ [5\binom{B+1}{B} + 5\binom{B+2}{B} + 3\binom{B+3}{B} + \binom{B+4}{B}]$$
$$= N_{B,B-3}^{B-1}(B+7).$$

$$N_{B,j}^{B-1}(B+7) = \binom{B+7}{j+4}[5\binom{B+1}{B} + 5\binom{B+2}{B} + 3\binom{B+3}{B} + \binom{B+4}{B}].$$
$$j=3,\ldots,B-4.$$

At distance d=B-3

$$N_{B,0}^{B-3}(B+7) = [4\binom{B+1}{B} + 4\binom{B+2}{B} + 3\binom{B+3}{B} + 2\binom{B+4}{B} + \binom{B+5}{B}]^2$$

$$N_{B,1}^{B-3}(B+7) = [\sum_{j=1}^{6}\binom{B+j}{B}][4\binom{B+1}{B}+4\binom{B+2}{B}+3\binom{B+3}{B}+2\binom{B+4}{B}+\binom{B+5}{B}]$$

$$N_{B,j}^{B-3}(B+7) = \binom{B+7}{j+5}[4\binom{B+1}{B}+4\binom{B+2}{B}+3\binom{B+3}{B}+2\binom{B+4}{B}+\binom{B+5}{B}].$$
$$j=2,\ldots,B-5.$$

At distance B-5

$$N_{B,0}^{B-5}(B+7) = [\sum_{j=1}^{6}\binom{B+j}{j}]^2 = N_{B,B-5}^{B-5}(B+7)$$

$$N_{B,j}^{B-5}(B+7) = [\sum_{j=1}^{6}\binom{B+j}{j}]\binom{B+7}{j+6}. \quad j=1,\ 2,\ldots,B-6.$$

At d < B-5

$$N_{B,j}^{d}(B+7) = \begin{cases} \binom{B+7}{k}\binom{B+7}{k+j} & \text{if } B=2k+d-7 \\ 0 & \text{otherwise} \end{cases}$$

At time t=B+8

at the boundary

$$N^{B}_{B,0}(B+8) = N^{B}_{B,B}(B+8) = [5\binom{B+1}{B}+5\binom{B+2}{B}+3\binom{B+3}{B}+\binom{B+4}{B}]^2$$

$$N^{B}_{B,1}(B+8) = [5\binom{B+1}{B}+5\binom{B+2}{B}+3\binom{B+3}{B}+\binom{B+4}{B}]$$

$$\cdot[9\binom{B+1}{B}+9\binom{B+2}{B}+6\binom{B+2}{B}+3\binom{B+4}{B}+\binom{B+5}{B}]$$

$$N^{B}_{B,2}(B+8) = [5\binom{B+1}{B}+5\binom{B+2}{B}+3\binom{B+3}{B}+\binom{B+4}{B}]$$

$$\cdot[5\binom{B+1}{B}+5\binom{B+2}{B}+4\binom{B+3}{B}+3\binom{B+4}{B}+2\binom{B+5}{B}+\binom{B+6}{B}]$$

$$N^{B}_{B,3}(B+8) = (\sum_{j=1}^{7}\binom{B+j}{B})[5\binom{B+1}{B}+5\binom{B+2}{B}+3\binom{B+3}{B}+\binom{B+4}{B}]$$

$$N^{B}_{B,j}(B+8) = \binom{B+8}{B}[5\binom{B+1}{B}+5\binom{B+2}{B}+3\binom{B+3}{B}+\binom{B+4}{B}]$$

$$j=4,\ldots,B-4.$$

At distance d=B-2

$$N^{B-2}_{B,0}(B+8) = [9\binom{B+1}{B}+9\binom{B+2}{B}+6\binom{B+3}{B}+3\binom{B+4}{B}+\binom{B+5}{B}]^2$$

$$N^{B-2}_{B,1}(B+8) = [9\binom{B+1}{B}+9\binom{B+2}{B}+6\binom{B+3}{B}+3\binom{B+4}{B}+\binom{B+5}{B}]$$

$$\cdot[5\binom{B+1}{B}+5\binom{B+2}{B}+4\binom{B+3}{B}+3\binom{B+4}{B}+2\binom{B+5}{B}+\binom{B+6}{B}]$$

$$N^{B-2}_{B,2}(B+8) = (\sum_{j=1}^{7}\binom{B+j}{B})[9\binom{B+1}{B}+9\binom{B+2}{B}+6\binom{B+3}{B}+3\binom{B+4}{B}+\binom{B+5}{B}]$$

$$N^{B-2}_{B,j}(B+8) = \binom{B+8}{B}[9\binom{B+1}{B}+9\binom{B+2}{B}+6\binom{B+3}{B}+3\binom{B+4}{B}+\binom{B+5}{B}]$$

At distance d=B-4

$$N_{B,0}^{B-4}(B+8) = [5\binom{B+1}{B}+5\binom{B+2}{B}+4\binom{B+3}{B}+3\binom{B+4}{B}+2\binom{B+5}{B}+\binom{B+6}{B}]^2$$

$$= N_{B,B-4}^{B-4}(B+8)$$

$$N_{B,1}^{B-4}(B+8) = [\sum_{k=1}^{7}\binom{B+k}{B}][5\binom{B+1}{B}+5\binom{B+2}{B}+4\binom{B+3}{B}+3\binom{B+4}{B}+2\binom{B+5}{B}+\binom{B+6}{B}]$$

$$N_{B,j}^{B-4}(B+8) = \binom{B+8}{j+6}[5\binom{B+1}{B}+5\binom{B+2}{B}+4\binom{B+3}{B}+3\binom{B+4}{B}+2\binom{B+5}{B}+\binom{B+6}{B}]$$

$$j=2,\ldots,B-6.$$

At distance d < B-4

$$N_j^d(B+8) = \begin{cases} \binom{B+8}{k}\binom{B+8}{k+j} & \text{if } B=2k+d-8 \\ 0 & \text{otherwise.} \end{cases}$$

This completes an analysis of the first eight time points past the time to the boundary. The expressions are quite unwieldy. However, the algorithm is clear and can compute message distribution throughout the grid at any time point. We write down the general set of equations and then indicate a "closed form" combinatorial method to make "sense" out of the unwieldy expressions.

VI.2 The General Equations

The initial conditions for t=B are:

At time t=B: Initial Conditions

$$N_{B,j}^{d}(B) = \begin{cases} \binom{B}{k}\binom{B}{k+j} & \text{if } B=2k+d \\ 0 & \text{otherwise} \end{cases}$$

The General Equations:

On the boundary $k \geq 1$

1. $N^{B}_{B,0}(B+2k) = N^{B}_{B,B}(B+2k) = N^{B-1}_{B,0}(B+2k-1)$.

2. $N^{B}_{B,j}(B+2k) = N^{B-1}_{B,j}(B+2k-1) + N^{B-1}_{B,j-1}(B+2k-1)$

$j=1, 2, \ldots, B-1.$

Note that:

$N^{B}_{B,j}(B+2k) = N^{B}_{B,B-j}(B+2k),$ $\quad j=0,\ldots,B.$

Along the axes $0 < d < B$, $k=1, 2, \ldots$

$$N^{d}_{B,0}(B+2k) = N^{d+1}_{B,0}(B+2k-1) + N^{d-1}_{B,0}(B+2k-1) + 2N^{d+1}_{B,1}(B+2k-1) = N^{d}_{B,d}(B+2k)$$

Off the axes $0 < d < B$, $0 < j < d$

$$N^{d}_{B,j}(B+2k) = N^{d+1}_{B,j}(B+2k-1) + N^{d-1}_{B,j}(B+2k-1) + N^{d-1}_{B,j-1}(B+2k-1) + N^{d+1}_{B,j+1}(B+2k-1).$$

At the origin:

$$N^{0}_{B,0}(B+2k) = 4N^{1}_{B,1}(B+2k-1).$$

This completes the general equations.

VI.3 A Conjecture on Solutions to the General Equation

An examination of the coefficients of the $\binom{B+k}{B}$ expressions in the first eight time points indicates that the coefficients

are the same as the rows in the following infinite sequence of tableaus.

T_1

	j=1	2	3	4	5	6
k=1	1					
k=2	1	1				
k=3	2	2	1			
k=4	5	5	3	1		
k=5	14	14	9	4	1	
k=6	42	42	28	14	5	1
	.	.	.	.	.	.
	.	.	.	.	.	.

T_2

k \ j	1	2	3	4	5	6	7
1	0	1					
2	1	1	1				
3	3	3	2	1			
4	9	9	6	3	1		
5	28	28	19	10	4	1	
6	90	90	62	34	15	5	1
	.	.	.	.	.	.	.
	.	.	.	.	.	.	.

T_3

k=1	0	0	1					
2	1	1	1	1				
3	4	4	3	2	1			
4	14	14	10	6	3	1		
5	48	48	34	20	10	4	1	
6	165	165	116	69	35	15	5	1
	.	.	.	.	.	.	.	.
	.	.	.	.	.	.	.	.

T_4

0	0	0	1					
1	1	1	1	1				
5	5	4	3	2	1			
20	20	15	10	6	3	1		
75	75	55	35	20	10	4	1	
275	275	200	125	70	35	15	5	1
.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.

The tableaus through T_4 cover all the coefficients which arise up to and including t=B+8. It is possible to give a solution for $S_{kj}(T_\nu)$ which is the j^{th} element in the k^{th} row of the tableau T_ν, $\nu = 1, 2, \ldots$ However, it is perhaps not obvious how these tableaus are generated. The j^{th} term of the k^{th} row of any tableau is obtained by summing the last $(k+\nu-j)$ terms in the previous row. That is, the third term in the fourth row of T_2 is obtained by summing the last three terms of the third row. In mathematical form the equation for the tableaus are:

Initial Condition $\quad S_{1,j}(T_\nu) = \begin{cases} 0, j<\nu, \\ 1, j=\nu \\ 0, j>\nu. \end{cases}$

$$S_{k,1}(T_\nu) = S_{k,2}(T_\nu) = \sum_{j=1}^{k+\nu-2} S_{k-1,j}(T_\nu).$$

$$S_{kj}(T_\nu) = \sum_{w=j-1}^{k+\nu-2} S_{k,w}(T_\nu), \qquad k>1$$

It is clear that the between tableau equations are:

$$S_{k,j}(T_\nu) = S_{k,j}(T_{\nu+1}) - S_{k-1,j}(T_{\nu+2}) \ , \ k>1, \ \nu = 1, 2,...$$

$$S_{k,j}(T_2) = S_{k+1,j}(T_1) - S_{k,j}(T_1).$$

Thus, to solve these equations it suffices to give a general formula for $S_{k,j}(T_1)$. The other terms can then be computed recursively. It is easy to check that the solution for T_1 is given by,

$$S_{k,j}(T_1) = \frac{j}{k}\binom{2k-j-1}{k-1}, \qquad \begin{matrix} j=1, 2,...,k \\ k=1, 2,...: \end{matrix}$$

Then by recursion:

$$S_{k,j}(T_2) = \frac{j}{k+1}\binom{2k-j+1}{k} - \frac{j}{k}\binom{2k-j-1}{k-1}, \qquad \begin{matrix} k=1, 2,.... \\ j=1, 2,...,k+1. \end{matrix}$$

$$S_{k,j}(T_3) = \frac{j}{k+2}\binom{2k-j+3}{k} + \frac{2j}{k+1}\binom{2k-j+1}{k}$$

$$\vdots$$

To apply these results we take a particular example to conjecture:

$$N^B_{B,0}(B+2k) = \left[\sum_{j=1}^{k} S_{k,j}(T_1)\binom{B+j}{B}\right]^2 \ , \qquad k \geq 1$$

$$= \left[\sum_{j=1}^{k} \frac{j}{k}\binom{2k-j-1}{k-1}\binom{B+j}{B}\right]^2$$

This conjecture holds for time points k=1, 2, 3, 4.

$$N_{B,0}^{B-1}(B+2k+1) = \left[\sum_{j=1}^{k+1} S_{k+1,j}\,(T_1)\binom{B+j}{B}\right]^2$$

This conjecture holds for all k=1, 2, 3, and satisfies the general equations.

VII. SUMMARY OF RESULTS

Recall that;

$N_o(t)$ = expected number of messages received at the origin at time t

$N'_o(t)$ = expected number of distinct messages received at the origin for the first time at time t.

$$Eff(t) = \frac{N_o(t)}{N'_o(t)}.$$

These symbols are in the model where there is Poisson input at each repeater at each time point. In all modes;

$$N'_o(t) = (2t^2+2t+1).$$

A. Deterministic The first model analyzed is the effect of a single message originally at the origin at time zero. No other messages ever enter the system. The propagation of this message can be measured by the following three quantities.

1) B(t) = the number of repeaters which receive the message for the 1st time at time t.

$$B(t) = 4t \text{ for } t \geq 1, \quad B(0) = 1$$

2) A(t) = the number of repeaters which have received the message by time t.

$$A(t) = 2t^2+2t+1.$$

3. $N_j^d(t)$ = number of copies of the message received at time t at a repeater with coordinates (d,j).

$$N_j^d(t) = 0, \quad \text{if} \quad t < d$$

$$N_j^d(d) = \binom{d}{j} \quad j=0, 1, \ldots, d; \quad d=0, 1, \ldots,$$

$$N_j^d(d+2k+1) = 0 \quad k=0, 1, 2, \ldots$$

$$N_j^d(d+2k) = \binom{d+2k}{k}\binom{d+2k}{k+j} \quad k=0, 1, \ldots$$

The quantity $N_j^d(d+2k)$ can be more generally interpreted as the number of copies of a message received at a repeater which is at distance d and horizontal distance j from the originating repeater, after d+2k time units.

B. Poisson Input Results

1. With no restrictions imposed on the operation of any repeater;

$$N_o(t) = \frac{\lambda}{3}(4^{t+1}-1)$$

$$N_o'(t) = \lambda(2t^2+2t+1)$$

$$\text{Eff}(t) = \frac{4^{t+1}-1}{3(2t^2+2t+1)} \sim \frac{4^{t+1}}{6t^2}$$

2. No Message Can Be Transmitted More Than k-Times

$$N_o(t) = \frac{\lambda}{3}[4^{k+1}-1]$$

$$N_o'(t) = \lambda[2k^2+2k+1]$$

$$\text{Eff}(t) = \frac{4^{k+1}-1}{3(2k^2+2k+1)}$$

Eff	1	2.60	3.40	8.3	22..
k	1	2	3	4	5

3. If the Same Message Arrives From Different Sources Only One Transmission is Made

This mode of operation assumes that a repeater can compare all messages arriving at the same time and transmit only one copy of duplicate messages. It is inherently assumed in this mode that there is only instantaneous memory (which we may call 0-memory)

$$N_o(2t) = \frac{(4t+3)(6t^2+1)}{3}$$

$$N'_o(2t) = (8t^2+4t+1)$$

$$Eff(2t) = \frac{(4t+3)(6t^2+1)}{3(8t^2+4t+1)} \sim \frac{4t}{3}$$

This mode produces a substantial reduction in duplicate traffic.

4. All Messages Are Transmitted in the Direction of a Fixed Ground Station

In this mode it is assumed that repeater "knows" where the message should be received.

$$N_o(t) = (2^{t+3}+4t-7)$$

$$N'_o(t) = (2t^2+2t+1)$$

$$Eff(t) = \frac{2^{t+3}+4t-7}{2t^2+2t+1} \sim \frac{2^{t+2}}{t^2}$$

This mode is better than no mode, but holds little promise by itself.

5. A Repeater Never Transmits The Same Message More Than Once, Except [illegible] Initial Reception

3. If the Same Message Arrives From Different Sources Only One Transmission is Made -

This mode of operation assumes that a repeater can compare all messages arriving at the same time and transmit only one copy of duplicate message. It is inherently assumed in this mode that there is only instantaneous memory (which we may call 0-memory).

$$N_0(2t) = \frac{(4t+3)(8t_2+1)}{3}$$

$$N_0'(2t) = (8t_2+4t+1)$$

$$Eff(2t) = \frac{(4t+3)(8t_2+1)}{3(8t_2+4t+1)} \rightarrow \frac{4t}{3}$$

This mode produces a substantial reduction in duplicate traffic.

4. All Messages Are Transmitted in the Direction of a Fixed Ground Station -

In this mode it is assumed that repeater "knows" where the message should be received.

$$N_0(t) = \frac{12t_2+4t-7}{}$$

$$N_0'(t) = (12t_2+2t+1)$$

$$Eff(t) = \frac{12t_2+4t-7}{2t_2+2t+1} \rightarrow \frac{2t+2}{t_2}$$

This mode is better than no mode, but holds little promise by itself.

5. A Repeater Never Transmits The Same Message More Than Once, Except Under Induced Rejection -

In this mode a repeater has "infinite" memory but not instantaneous memory.

$$N_o(t) = \lambda(2^{t+4}-16t-11)$$

$$N'_o(t) = \lambda(2t^2+2t+1)$$

$$Eff(t) = \frac{2^{t+4}-16t-11}{2t^2+2t+1} \sim \frac{2^{t+3}}{t^2}$$

It appears that infinite memory without instant memory does not "damp" the message flow significantly: in fact it is worse than "directionality" in the repeater.

6. Infinite Memory Plus Instant Memory

If we combine the two modes of operation

$$N_o(t) = \lambda(8t^2-8t-3)$$

$$N'_o(t) = \lambda(2t^2+2t+1)$$

$$Eff(t) = \frac{8t^2-8t-3}{2t^2+2t+1} \sim 4$$

The effect of combining the instant memory feature with the instantaneous memory feature is to reduce the message flow in the instantaneous mode by a factor of t/3 which is significant.

7. Instant Memory Plus no Message Transmitted More Than k-Times

In this mixed mode the analysis shows that all results in the instant memory case are the same except that the flow becomes independent of time after $t \geq k$. The efficiency is

$$Eff(t) \sim \frac{4}{3}k$$

k	1	2	3	4	5	6
Eff	1.33	2.67	4	5.33	6.67	8